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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 752

Spatial Triangulation by Least Squares Adjustment of Conditioned Observations

HELLMUT SCHMID

ABERDEEN PROVING GROUND, MARYLAND

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SPATIAL TRIANGULATION BY LEAST SQUARES ADJUSTMENT OF CONDITIONED OBSERVATIONS

Hellmut Schmid

Project No. TB3-0838 of the Research and Development Division, Ordnance Corps

ABERDEEN PROVING GROUND, MARYLAND

TABLE OF CONTENTS

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ABSTRACT

A rigid method of least squares adjustment for a spatial triangulation from angle measurements is outlined. The method is based on an adjustment of direct observations which are conditioned. The results are derived for Cartesian, rectangular spherical, and rectangular ellipsoidal coordinates.

STATEMENT OF PROBLEM

The present report contains an analysis of the problem of determining the most probable position in space of a point, given the azimuth and elevation angles to the point measured with theodolites at various stations.

INTRODUCTION

Because of errors in the measured angles it is unlikely that any two lines of sight will intersect in space. Even after adjustment of the instruments (determination of the systematic errors) and after application of corrections to the instrumental readings according to those systematic errors, there will be no spatial intersections because of unavoidable accidental errors, which may be explained as a composite of remaining instrumental and operational errors. Therefore, it is necessary to determine a point in space as the most probable point.

The criterion for the most probable point is given by the principle of a rigid least squares adjustment; namely, that the sum of the squares of the corrections which must be applied on the original measurements becomes a minimum. This means that in the present case we must find that point in space for which the sum of the squares of the angular (azimuth and elevation) deviations between the observed lines of sight and the corresponding most probable lines of sight for all observation stations becomes a minimum.

Especially if the measurements are made by different types of measuring instruments, the observed azimuth and elevation angles are not necessarily obtained with the same accuracy or in other words with the same weight. Therefore, a rigid adjustment must provide the possibility of introducing weighting factors.

VARIOUS METHODS OF LEAST SQUARES ADJUSTMENT

The purpose of a least squares adjustment is the determination of the most probable values for the desired quantities by compensating for the differences resulting from unavoidable accidental errors. There are several types of least squares adjustments, depending upon the measuring method and the nature of the measured quantities. We may have cases in which the desired quantity can be measured either directly or indirectly, and where the observations are either independent or dependent. Dependent observations are those for which a mathematical condition exists between the observations. Accordingly, we have four different types of least squares adjustment:

1. Adjustment of independent direct observations.

2. Adjustment of independent indirect observations.

3. Adjustment of direct observations which are dependent or conditioned.

4. Adjustment of indirect observations with condition equations.

PROPOSED LEAST SQUARES ADJUSTMENT

The spatial triangulation of a point can be solved either by introducing the spatial coordinates of the desired point (x, y, z) as unknowns and by applying an adjustment according to the principle of (2), or by considering the most probable values of the measured angles as unknowns and adjusting these angles according to (3), as direct observations, with condition equations.

In any least squares solution the main part of computing is the solution of the normal equations. To decide upon the method means mostly to select the method with the smallest number of normal equations. However, further consideration must be given to the coefficients in the observation equations, especially with regard to the required number of digits. In the adjustment of independent indirect observations the number of normal equations equals the number of unknowns. In our case, with the unknowns (x, y, z), we would have three normal equations. The observation equations will not be linear initially, but must be linearized by means of the Taylor Series. This step calls for approximation values of the unknowns. The required accuracy of the coefficients can be expected to be relatively high, because angular deviations cause different displacements on the coordinates depending on the geometrical configuration of the triangulation case.

In the adjustment of conditioned observations the number of normal equations equals the number of condition equations. The number of condition equations equals the number of observations which are in excess.

Because three angles are necessary to provide a rigorous solution, we have for:

Number of Stations	2	3	4	5	n
Number of Observations	4	6	8	10	2n
Number of Angles Necessary for a Rigid Solution	3	3	3	3	3
Number of Normal Equations	1	3	5	7	2n-3

Because of the fact that the angular corrections are small and of the same order of magnitude, independent of the absolute value of the measured angle, the coefficients in the observation equations need carry only a few digits.

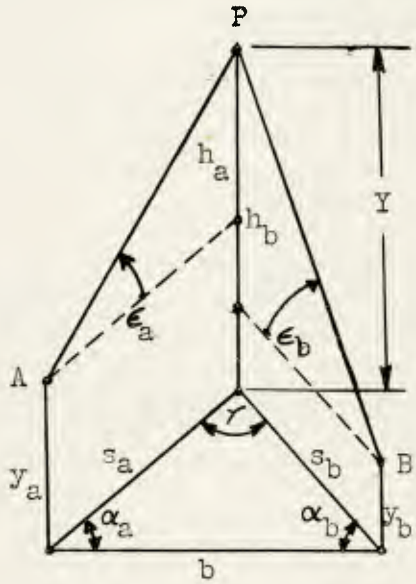
The selection of the most economical least squares adjustment may be based on the above-mentioned facts.

It appears that because of the simplicity of the coefficients in the observation equations, the adjustment with conditioned observations will be advantageous for three, and possibly four, measuring stations. For five or more stations the method of independent indirect observations

should be applied. In missile trajectory measurement the majority of missile points are determined from two or three measuring stations. The case in which more than four stations are in such a favorable geometrical pattern that they will be used for a single intersection is so infrequent that it may be treated by special methods such as grouping of the stations. Therefore, in all practical cases of missile trajectory measurement the method of adjustment of direct conditioned observations will be useful.

DERIVATION OF THE SOLUTION

1. For a Cartesian coordinate system:



Coordinates of the camera stations are:

Station	X	Z	Y
A	Xa	Za	Ya
B	Xb	Zb	Yb
C	Xc	Zc	Yc
N	Xn	Zn	Yn

The measured angles at each station corrected for instrumental systematic errors are:

Azimuth : α_i

Elevation: ϵ_i

The elevation angles are assumed to be corrected for refraction.

The first step is to establish the condition equation. For two stations with four observations, we have 4-3 = 1 condition equation. This condition equation can be obtained if the elevation Y, of P, is expressed first from a in terms of α_a, S_a, Y_a, ϵ_a and second, from b in terms of α_b, S_b, Y_b, ϵ_b.

$$Y = h_a + y_a = \frac{b \sin\alpha_b \tan\epsilon_b}{\sin(\alpha_a + \alpha_b)} + y_a = h_b + y_b = \frac{b \sin\alpha_a \tan\epsilon_b}{\sin(\alpha_a + \alpha_b)} + y_b$$

$$h_a = S_a \tan\epsilon_a \qquad h_b = S_b \tan\epsilon_b$$

$$S_a = \frac{b \sin\alpha_b}{\sin(\alpha_a + \alpha_b)} \qquad S_b = \frac{b \sin\alpha_a}{\sin(\alpha_a + \alpha_b)}$$

and the rigorous condition equation:

$$\sin\alpha_b \tan\epsilon_a - \sin\alpha_a \tan\epsilon_b + a \sin(\alpha_a + \alpha_b) = 0 \qquad (1)$$

where $a = \frac{\Delta y}{b}$ $\quad \Delta y = y_a - y_b$ $\quad$ and $\quad b$ = base

If the observed angles are inserted in the condition equation (1), the equation will not reduce to zero but leave a discrepancy d.

If v_{α_a}, v_{α_b}, v_{ϵ_a} and v_{ϵ_b} denote the corrections which will reduce the measured angles to the most probable values, the condition equation becomes:

$$F = \sin(\alpha_b + v_{\alpha_b}) \tan(\epsilon_a + v_{\epsilon_a}) - \sin(\alpha_a + v_{\alpha_a}) \tan(\epsilon_b + v_{\epsilon_b}) +$$

$$a \sin(\alpha_a + v_{\alpha_a} + \alpha_b + v_{\alpha_b}) = 0 \qquad (2)$$

To linearize this equation we have, by Taylor's Series, neglecting the terms of second and higher power:

$$d + \left(\frac{\partial F}{\partial \alpha_a}\right) \cdot \frac{v_{\alpha_a}}{\rho} + \frac{\partial F}{\partial \alpha_b} \cdot \frac{v_{\alpha_b}}{\rho} + \frac{\partial F}{\partial \epsilon_a} \cdot \frac{v_{\epsilon_a}}{\rho} + \frac{\partial F}{\partial \epsilon_b} \frac{v_{\epsilon_b}}{\rho} = 0$$

or:

$$a_1 \cdot v_{\alpha_a} + a_2 v_{\alpha_b} + a_3 v_{\epsilon_a} + a_4 v_{\epsilon_b} + d' = 0 \qquad (3)$$

where $\gamma = 180 - (\alpha + \beta)$

$$a_1 = -(\cos\alpha_a \tan\epsilon_b + a \cos\gamma)$$

$$a_2 = +(\cos\alpha_b \tan\epsilon_a - a \cos\gamma)$$

$$a_3 = + \sin\alpha_b \sec^2\epsilon_a$$

$$a_4 = - \sin\alpha_a \sec^2\epsilon_b$$

$$d' = \rho \cdot d = \rho(\sin\alpha_b \tan\epsilon_a - \sin\alpha_a \tan\epsilon_b + a \sin\gamma)$$

The least squares computation requires now determining the corrections v_{α_a}, v_{α_b}, v_{ϵ_a}, v_{ϵ_b}, so that the condition equation (3) is satisfied and that, for observations with equal weight, $[vv]$ or for observations of different weight, $[pvv]$ becomes a minimum.

Introducing a new parameter λ, a new function f can be derived from (3) as follows:

$$f = p_{\alpha_a} \cdot v_{\alpha_a}^2 + p_{\alpha_b} \cdot v_{\alpha_b}^2 + p_{\epsilon_a} \cdot v_{\epsilon_a}^2 + p_{\epsilon_b} \cdot v_{\epsilon_b}^2$$

$$+ \lambda (a_1 \, v_{\alpha_a} + a_2 \, v_{\alpha_b} + a_3 \, v_{\epsilon_a} + a_4 \, v_{\epsilon_b} + d')$$

or, in another arrangement:

$$\begin{aligned} f = &+ p_{\alpha_a} \, v_{\alpha_a}^2 + v_{\alpha_a} \, a_1 \lambda \\ &+ p_{\alpha_b} \, v_{\alpha_b}^2 + v_{\alpha_b} \, a_2 \lambda \\ &+ p_{\epsilon_a} \, v_{\epsilon_a}^2 + v_{\epsilon_a} \, a_3 \lambda \\ &+ p_{\epsilon_b} \, v_{\epsilon_b}^2 + v_{\epsilon_b} \, a_4 \lambda \\ &+ d' \lambda \end{aligned}$$

To determine the five unknowns v_{α_a}, v_{α_b}, v_{ϵ_a}, v_{ϵ_b} and λ

we have the five equations:

$$\begin{aligned} \frac{\partial f}{\partial v_{\alpha_a}} &= 2 \, p_{\alpha_a} \cdot v_{\alpha_a} + a_1 \lambda \\ \frac{\partial f}{\partial v_{\alpha_b}} &= 2 \, p_{\alpha_b} \cdot v_{\alpha_b} + a_2 \lambda \\ \frac{\partial f}{\partial v_{\epsilon_a}} &= 2 \, p_{\epsilon_a} \cdot v_{\epsilon_a} + a_3 \lambda \\ \frac{\partial f}{\partial v_{\epsilon_b}} &= 2 \cdot p_{\epsilon_b} \cdot v_{\epsilon_b} + a_4 \lambda \\ a_1 \cdot v_{\alpha_a} &+ a_2 \, v_{\alpha_b} + a_3 \, v_{\epsilon_a} + a_4 \, v_{\epsilon_b} + d' = 0 \end{aligned} \tag{4}$$

Introducing $\lambda = -2K$

$$v_{\alpha_a} = \frac{a_1}{p_{\alpha_a}} K$$

$$v_{\alpha_b} = \frac{a_2}{p_{\alpha_b}} K$$

$$v_{\epsilon_a} = \frac{a_3}{p_{\epsilon_a}} K \qquad (5)$$

$$v_{\epsilon_b} = \frac{a_4}{p_{\epsilon_b}} K$$

By substitution of these values in the condition equation (3) we obtain the normal equation:

$$\left[\frac{aa}{p}\right] K + d' = 0 \quad \text{or} \quad k = - \frac{d'}{\left[\frac{aa}{p}\right]}$$

With the help of the correlate K, the corrections v can be calculated from (5) and thus the most probable values of the measured angles are determined. A partial check for the calculation of the corrections v is obtained from:

$$[pvv] = - [d'K]$$

The mean error of an observation of unit weight is

$$\mu_o = \pm \sqrt{\frac{[pvv]}{\text{number of observations in excess}}}$$

and the mean error of an observation of weight p_i is $\mu_i = \pm \frac{\mu_o}{\sqrt{p_i}}$

If there are more than two stations, each new station may be combined with each of the old ones, thus giving rise to additional condition equations. As an example for three stations A, B, C with six independent observations, we have 6-3 = 3 condition equations which are, according to (3):

$$a_1 \, v_{\alpha_a} + a_2 \, v_{\alpha_b} + a_4 \, v_{\epsilon_a} + a_5 \, v_{\epsilon_b} + d'_1 = 0$$

$$b_2 \, v_{\alpha_b} + b_3 \, v_{\alpha_c} + b_5 \, v_{\epsilon_b} + b_6 \, v_{\epsilon_c} + d'_2 = 0 \qquad (3')$$

$$c_1 \, v_{\alpha_a} + c_3 \, v_{\alpha_c} + c_4 \, v_{\epsilon_a} + c_6 \, v_{\epsilon_c} + d'_3 = 0$$

Equation (4) then becomes:

$$
\begin{aligned}
2p_{\alpha_a} v_{\alpha_a} + a_1 \lambda_1 + c_1 \lambda_3 &= 0 \\
2p_{\alpha_b} v_{\alpha_b} + a_2 \lambda_1 + b_2 \lambda_2 &= 0 \\
2p_{\alpha_c} v_{\alpha_c} + b_3 \lambda_2 + c_3 \lambda_3 &= 0 \\
2p_{\epsilon_a} v_{\epsilon_a} + a_4 \lambda_1 + c_4 \lambda_3 &= 0 \\
2p_{\epsilon_b} v_{\epsilon_b} + a_5 \lambda_1 + b_5 \lambda_2 &= 0 \\
2p_{\epsilon_c} v_{\epsilon_c} + b_6 \lambda_2 + c_6 \lambda_3 &= 0
\end{aligned}
\tag{4'}
$$

and, for $\lambda_1 = -2K_1$ $\quad \lambda_2 = -2K_2$ $\quad \lambda_3 = -2K_3$

$$
\begin{aligned}
v_{\alpha_a} &= \frac{a_1 K_1 + c_1 K_3}{p_{\alpha_a}} \\
v_{\alpha_b} &= \frac{a_2 K_1 + b_2 K_2}{p_{\alpha_b}} \\
v_{\alpha_c} &= \frac{b_3 K_2 + c_3 K_3}{p_{\alpha_c}} \\
v_{\epsilon_a} &= \frac{a_4 K_1 + c_4 K_3}{p_{\epsilon_a}} \\
v_{\epsilon_b} &= \frac{a_5 K_1 + b_5 K_2}{p_{\epsilon_b}} \\
v_{\epsilon_c} &= \frac{b_6 K_2 + c_6 K_3}{p_{\epsilon_c}}
\end{aligned}
\tag{5'}
$$

Substituting these values in (3') we obtain the normal equations:

$$\left[\frac{aa}{p}\right] K_1 + \left[\frac{ab}{p}\right] K_2 + \left[\frac{ac}{p}\right] K_3 + d'_1 = 0$$

$$\left[\frac{ab}{p}\right] K_1 + \left[\frac{bb}{p}\right] K_2 + \left[\frac{bc}{p}\right] K_3 + d'_2 = 0$$

$$\left[\frac{ac}{p}\right] K_1 + \left[\frac{bc}{p}\right] K_2 + \left[\frac{cc}{p}\right] K_3 + d'_3 = 0$$

By means of the derived correlates, K_1, K_2, K_3 and (5'), the corrections v may then be determined

Numerical example for two stations A and B

Measured angles: $\alpha_a = 38^\circ 24' 10''$

$\alpha_b = 38\ 25\ 50$

$\epsilon_a = 9\ 06\ 00$

$\epsilon_b = 9\ 43\ 50$

$\gamma = \alpha_a + \alpha_b = 76\ 50\ 00$

Length of the base AB = 54 614.89.
Difference in elevation of the two stations,

$\Delta Y = +\ 393.80$

$a = \frac{\Delta y}{b} = 0.00721049$

The elevation angles have been corrected for refraction:

$\sin \alpha_a = 0.62186$	$\cos \alpha_a = 0.78$	
$\sin \alpha_b = 0.621566$	$\cos \alpha_b = 0.78$	
$\sin \gamma = 0.973712$	$\cos \gamma = 0.23$	
$\tan \epsilon_a = 0.160174$	$\cos \epsilon_a = 0.99$	$\cos^2 \epsilon_a = 0.97$
$\tan \epsilon_b = 0.171482$	$\cos \epsilon_b = 0.99$	$\cos^2 \epsilon_b = 0.97$

$a_1 = -\ 0.13$

$a_2 = +\ 0.12$

$a_3 = +\ 0.64$

$a_4 = -\ 0.64$ $\qquad [aa] = 0.85 \qquad K = -\ 13.9$

$d' = +\ 0.000057 \cdot \rho = +\ 11.8$

$v_{\alpha_a} = +1.8''$ Result: $\alpha_a + v_{\alpha_a} = 38°24'12''$ Check:

$v_{\alpha_b} = -1.7$ $\alpha_b + v_{\alpha_b} = 38\ 25\ 48$ $[vv] = +164.6$

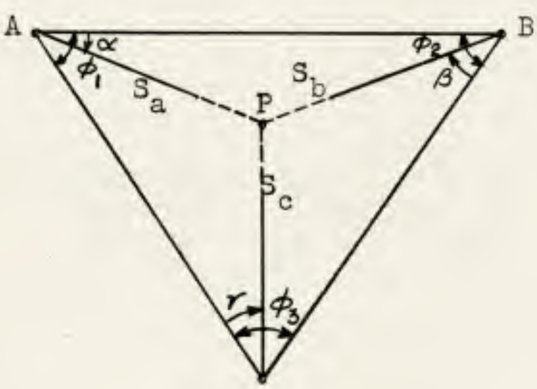
$v_{\epsilon_a} = -8.9$ $\epsilon_a + v_{\epsilon_a} = 9\ 05\ 51$ $-[Kd] = +164.0$

$v_{\epsilon_b} = +8.9$ $\epsilon_b + v_{\epsilon_b} = 9\ 43\ 59$

It is noted that all computations for the coefficients a_1 to a_4 carry only two digits. Special cases of the solution occur if not all angles have been measured.

If we have, for example, three stations, it may happen that all three azimuth angles are available but only one elevation angle. With these four measured quantities we obtain 4-3 = 1 condition equation and therefore one normal equation.

The condition equation can be found by means of the following computations:



$$S_b = \frac{S_a \sin\alpha}{\sin(\phi_2 - \beta)}$$

$$S_c = \frac{S_b \sin\beta}{\sin(\phi_3 - \gamma)}$$

$$S_a = \frac{S_c \sin\gamma}{\sin(\phi_1 - \alpha)} \qquad \text{or}$$

$$\frac{\sin\alpha \sin\beta \sin\gamma}{\sin(\phi_1 - \alpha)\sin(\phi_2 - \beta)\sin(\phi_3 - \gamma)} = 1 \tag{6}$$

Introducing

$$\alpha' = \alpha + v_\alpha$$
$$\beta' = \beta + v_\beta$$
$$\gamma' = \gamma + v_\gamma$$

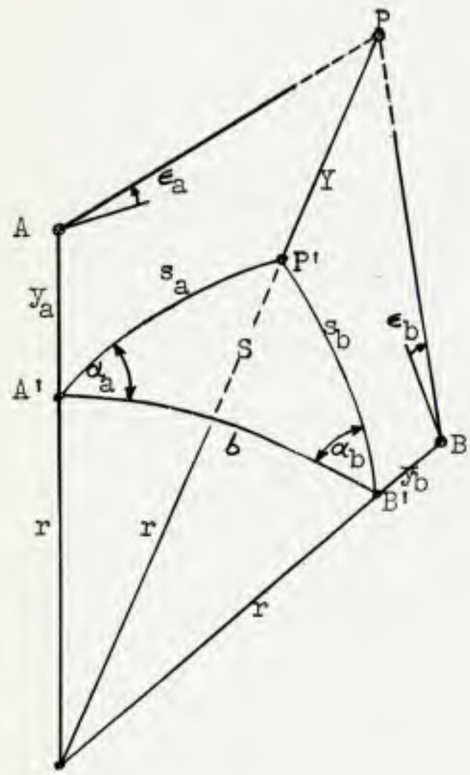
by means of Taylor's Series, neglecting the terms of second and higher order, we obtain:

$$+ \cos\alpha\sin\beta\sin\gamma + \cos(\phi_1 - \alpha)\sin(\phi_2 - \beta)\sin(\phi_3 - \gamma)$$
$$+ \sin\alpha\cos\beta\sin\gamma + \sin(\phi_1 - \alpha)\cos(\phi_2 - \beta)\sin(\phi_3 - \gamma)$$
$$+ \sin\alpha\sin\beta\cos\gamma + \sin(\phi_1 - \alpha)\sin(\phi_2 - \beta)\cos(\phi_3 - \gamma)$$
$$+ \sin\alpha\sin\beta\sin\gamma - \sin(\phi_1 - \alpha)\sin(\phi_2 - \beta)\sin(\phi_3 - \gamma) = 0$$

or $a'_1 v_\alpha + a'_2 v_\beta + a'_3 v_\gamma + D = 0$ (7)

To complete the analysis the case should be considered where all three elevation angles were observed but only one azimuth angle. This case cannot be expressed in a simple condition equation. The practical computation of the coordinates of the missile position requires the use of approximate values of the unknown coordinates, thus providing the means for an adjustment of independent indirect observations.

2. <u>Solution for a spherical rectangular coordinate system.</u>



The measuring stations A and B have the spherical rectangular coordinates:

$$x_a,\ z_a,\ y_a$$

and $$x_b,\ z_b,\ y_b$$

The measured angles are:

Station	Azimuth	Elevation
A	α_a	ϵ_a
B	α_b	ϵ_b

The elevation angles are assumed to have been corrected for refraction. The length of the baseline AB is denoted by b.

$$Y = y_a + a_1 s_a \tan\epsilon_a + a_1 s_a \left[\frac{s_a}{r}(0.5 + \tan^2\epsilon_a) + \left(\frac{s_a}{r}\right)^2 \tan\epsilon_a \left(\frac{5}{6} + \tan^2\epsilon_a\right) + \left(\frac{s_a}{r}\right)^3 \left(\frac{5}{24} + \frac{7}{6}\tan^2\epsilon_a + \tan^4\epsilon_a\right)\right]$$

$$Y = y_b + a_2 s_b \tan\epsilon_b + a_2 s_b \left[\frac{s_b}{r}(0.5 + \tan^2\epsilon_b) + \left(\frac{s_b}{r}\right)^2 \tan\epsilon_b \left(\frac{5}{6} + \tan^2\epsilon_b\right) + \left(\frac{s_b}{r}\right)^3 \left(\frac{5}{24} + \frac{7}{6}\tan^2\epsilon_b + \tan^4\epsilon_b\right)\right]$$

$$a_1 = \left(1 + \frac{y_a}{r}\right) \qquad \Delta y = y_a - y_b$$

$$a_2 = \left(1 + \frac{y_b}{r}\right) \qquad a_3 = \frac{\Delta y}{b} \qquad \rho = 206\ 264''.8062$$

$$a_1 s_a \tan\epsilon_a - a_2 s_b \tan\epsilon_b + \Delta y + a_1 s_a [I] - a_2 s_b [II] = 0 \tag{8}$$

The spherical excess of the triangle A'B'P' is $\Delta = \frac{S}{r^2}\rho = f(\alpha_a, \alpha_b, b)$

$$\frac{\Delta}{3} = \frac{b^2\rho}{6r^2} \cdot \frac{\sin\alpha_a \sin\alpha_b}{\sin(\alpha_a+\alpha_b)} = K \cdot \frac{\sin\alpha_a \sin\alpha_b}{\sin(\alpha_a+\alpha_b)}$$

The values for $\frac{\Delta}{3}$ can be tabulated for a given baseline with the two arguments α_a and α_b:

$$\alpha'_a = \alpha_a - \frac{\Delta}{3} \qquad s_a = \frac{\sin\alpha'_b}{\sin\gamma'} \cdot b$$

$$\alpha'_b = \alpha_b - \frac{\Delta}{3} \qquad \text{(Set of Legendre)}$$

$$\gamma' = \alpha'_a + \alpha'_b = \alpha_a + \alpha_b - \frac{2\Delta}{3} \qquad s_b = \frac{\sin\alpha'_a}{\sin\gamma'} \cdot b$$

Substituting s_a and s_b in (8) gives:

$$a_1 \sin\alpha'_b \tan\epsilon_a - a_2 \sin\alpha'_a \tan\epsilon_b + a_3 \sin\gamma' + a_1 \sin\alpha'_b [I] - a_2 \sin\alpha'_2 [II] = 0 \tag{9}$$

The most probable values of the measured angles are:

$$\begin{aligned} (\alpha_a) &= \alpha_a + v_{\alpha_a} \\ (\alpha_b) &= \alpha_b + v_{\alpha_b} \\ (\epsilon_a) &= \epsilon_a + v_{\epsilon_a} \\ (\epsilon_b) &= \epsilon_b + v_{\epsilon_b} \end{aligned} \tag{10}$$

Inserting the measured angles in the condition equation (9) we obtain a difference d, caused by accidental errors in the measured angles.

$$F = a_1 \sin\alpha'_b (\tan\epsilon_a + [I]) - a_2 \sin\alpha'_a (\tan\epsilon_b + [II]) + a_3 \sin\gamma' = d \tag{11}$$

Using Taylor's Series, and neglecting terms of second and higher order, we obtain from (9) and (10):

$$d \cdot \rho + \frac{\partial F}{\partial\alpha_a} v_{\alpha_a} + \frac{\partial F}{\partial\alpha_b} v_{\alpha_b} + \frac{\partial F}{\partial\epsilon_a} v_{\epsilon_a} + \frac{\partial F}{\partial\epsilon_b} v_{\epsilon_b} = 0 \qquad \text{or:}$$

$$A_1 v_{\alpha_a} + A_2 v_{\alpha_b} + A_3 v_{\epsilon_a} + A_4 v_{\epsilon_b} + d' = 0$$

The determination of v_{α_a}, v_{α_b}, v_{ϵ_a}, and v_{ϵ_b} follows the procedure outlined in the solution for a Cartesian system, and gives:

$$v_{\alpha_a} = A_1 \cdot K \qquad\qquad K = -\frac{d}{[AA]}$$

$$v_{\alpha_b} = A_2 \cdot K$$

$$v_{\epsilon_a} = A_3 \cdot K \qquad\qquad \text{Check:}\quad [vv] = -[Kd]$$

$$v_{\epsilon_b} = A_4 \cdot K$$

$$A_1 = \frac{\partial F}{\partial \alpha_a} = -a_2 \cos\alpha'_a \,(\tan\epsilon_b + [II]) + a_3 \cos\gamma'$$

$$- a_1 \sin\alpha'_b \cot\gamma' \Big[\frac{s_a}{r}\left(\tfrac{1}{2} + \tan^2\epsilon_a\right) + 2\left(\frac{s_a}{r}\right)^2 \tan\epsilon_a \left(\tfrac{5}{6} + \tan^2\epsilon_a\right) + 3\left(\frac{s_a}{r}\right)^3 \left(\tfrac{5}{24} + \tfrac{7}{6}\tan^2\epsilon_a + \tan^4\epsilon_a\right)\Big]$$

$$- a_2 \sin\alpha'_a \operatorname{cosec}\gamma' \Big[\frac{s_a}{r}\left(\tfrac{1}{2} + \tan^2\epsilon_b\right) + 2\,\frac{s_a s_b}{r^2} \tan\epsilon_b \left(\tfrac{5}{6} + \tan^2\epsilon_b\right) + 3\,\frac{s_a^2 s_b}{r^3}\left(\tfrac{5}{24} + \tfrac{7}{6}\tan^2\epsilon_b + \tan^4\epsilon_b\right)\Big]$$

$$A_2 = \frac{\partial F}{\partial \alpha_b} = +a_1 \cos\alpha'_b \,(\tan\epsilon_a + [I]) + a_3 \cos\gamma'$$

$$+ a_1 \sin\alpha'_b \operatorname{cosec}\gamma' \Big[\frac{s_b}{r}\left(\tfrac{1}{2} + \tan^2\epsilon_a\right) + 2\,\frac{s_a s_b}{r^2} \tan\epsilon_a \left(\tfrac{5}{6} + \tan^2\epsilon_a\right) + 3\,\frac{s_a^2 s_b}{r^3}\left(\tfrac{5}{24} + \tfrac{7}{6}\tan^2\epsilon_a + \tan^4\epsilon_a\right)\Big]$$

$$+ a_2 \sin\alpha'_a \cot\gamma' \Big[\frac{s_b}{r}\left(\tfrac{1}{2} + \tan^2\epsilon_b\right) + 2\left(\frac{s_b}{r}\right)^2 \tan\epsilon_b \left(\tfrac{5}{6} + \tan^2\epsilon_b\right) + 3\left(\frac{s_b}{r}\right)^3 \left(\tfrac{5}{24} + \tfrac{7}{6}\tan^2\epsilon_b + \tan^4\epsilon_b\right)\Big]$$

$$A_3 = \frac{\partial F}{\partial \epsilon_a} = +\frac{a_1 \sin\alpha'_b}{\cos^2\epsilon_a}\Big[1 + \left(\frac{s_a}{r}\right) 2\tan\epsilon_a + \left(\frac{s_a}{r}\right)^2\left(\tfrac{5}{6} + 3\tan^2\epsilon_a\right) + \left(\frac{s_a}{r}\right)^3\left(\tfrac{7}{3}\tan\epsilon_a + 4\tan^3\epsilon_a\right)\Big]$$

$$A_4 = \frac{\partial F}{\partial \epsilon_b} = -\frac{a_2 \sin\alpha'_a}{\cos^2\epsilon_b}\left[1+ (\frac{s_b}{r})\, 2\tan\epsilon_b + (\frac{s_b}{r})^2 (\frac{5}{6}+3\tan^2\epsilon_b) + (\frac{s_b}{r})^3 (\frac{7}{3}\tan\epsilon_b + 4\tan^3\epsilon_b)\right]$$

Because of the fact that the coefficients $A_1 - A_4$ must be computed with a few digits only, considerable simplifications are possible. The following formulas may be considered adequate for ranges up to 150 miles and elevation angles up to 80°, even for high precision instrumentation. In the following formulas the number of terms to be carried depends on the geometry involved in the triangulation case and on the accuracy of the instruments used.

The difference d is calculated from (11) as follows:

$$d = a_1 \sin\alpha'_b (\tan\epsilon_a + [I]) - a_2 \sin\alpha'_a (\tan\epsilon_b + [II]) + a_3 \sin\gamma'$$

$$a_1 = 1 + \frac{y_a}{r} \qquad a_2 = 1 + \frac{y_b}{r} \qquad a_3 = \frac{\Delta y}{b} = \frac{y_a - y_b}{b}$$

$$\alpha'_a = \alpha_a - \frac{\Delta}{3} \qquad \alpha'_b = \alpha_b - \frac{\Delta}{3} \qquad \gamma' = \alpha_a' + \alpha_b' \qquad \frac{\Delta}{3} = K\frac{\sin\alpha_a \sin\alpha_b}{\sin(\alpha_a + \alpha_b)}$$

$$s_a = \frac{b \sin\alpha'_b}{\sin\gamma'} \qquad s_b = \frac{b\sin\alpha'_a}{\sin\gamma'} \qquad K = \frac{b^2\rho}{6r^2};\ \rho = 206\ 265''$$

$\frac{\Delta}{3}$ can be tabulated

$$[I] = \frac{s_a}{r}(\frac{1}{2}+\tan^2\epsilon_a) + (\frac{s_a}{r})^2 \tan\epsilon_a (\frac{5}{6}+\tan^2\epsilon_a) + (\frac{s_a}{r})^3 (\frac{5}{24}+\frac{7}{6}\tan^2\epsilon_a + \tan^4\epsilon_a)$$

$$[II] = \frac{s_b}{r}(\frac{1}{2}+\tan^2\epsilon_b) + (\frac{s_b}{r})^2 \tan\epsilon_b (\frac{5}{6}+\tan^2\epsilon_b) + (\frac{s_b}{r})^3 (\frac{5}{24}+\frac{7}{6}\tan^2\epsilon_b + \tan^4\epsilon_b)$$

$$A_1 = -a_2 \cos\alpha'_a (\tan\epsilon_b + [II]) + a_3 \cos\gamma'$$

$$A_2 = +a_1 \cos\alpha'_b (\tan\epsilon_a + [I]) + a_3 \cos\gamma'$$

$a_3 \cos\gamma'$ is only significant if the difference in elevation between the measuring stations is considerable

$$A_3 = +\frac{a_1 \sin\alpha'_b}{\cos^2\epsilon_a}(1 + 2\frac{s_a}{r}\tan\epsilon_a)$$

$$A_4 = -\frac{a_2 \sin\alpha'_a}{\cos^2\epsilon_b}(1 + 2\frac{s_b}{r}\tan\epsilon_b)$$

In case all the elevation angles are not available the condition equation (7) may be used. The angles must be changed first to plane angles by subtracting one third of the corresponding spherical excess.

3. Solution for an ellipsoidal rectangular coordinate system

The radius of curvature for a point on the ellipsoidal earth changes with the azimuth. If M is the radius in the meridian and N the radius perpendicular to the meridian, there is the relation $\frac{N}{M} = V^2$. The more nearly this ratio is equal to one, the more nearly may the earth be considered as a sphere in the vicinity of the point.

ϕ Latitude	$\frac{N}{M} = V^2$
0°	1,007
30°	1,005
45°	1,003
60°	1,002
90°	1,000

The values V^2 are rather different from 1 and only on the poles ($\phi = 90^\circ$) does V^2 become equal to 1. For a latitude of 30° we have a deviation of about 1/2%.

Because we may neglect for our purpose the change of curvature with the change of the mean latitude of the triangulation side involved, it is possible to substitute for the ellipsoid at a measuring station a number of spheres whose radii change with the azimuth of the considered sides. The radii can be tabulated for a station at suitable intervals. The ellipsoidal calculation may now use the same formulas which were derived in the chapter on a spherical coordinate system. The only difference is that instead of a single radius for all measuring stations a specific radius for each line of sight must be introduced.

NUMERICAL EXAMPLE FOR THREE STATIONS IN AN ELLIPSOIDAL RECTANGULAR COORDINATE SYSTEM WITH DIFFERENT TYPES OF INSTRUMENTS AND INCOMPLETE ANGLE MEASUREMENTS

The ellipsoidal rectangular coordinates of the measuring stations, expressed in meters, are:

Station	x	z	y
N	+ 37010.871	- 31383.845	1524.003
P	- 11088.230	0	1219.202
O	+ 92519.561	+ 46815.110	2133.604

The length of the baseline $\overline{NP}$ is b = 57432.138

$$\Delta y = y_n - y_p = + 304.801 \qquad a_3 = \frac{\Delta y}{b} = +.005307$$

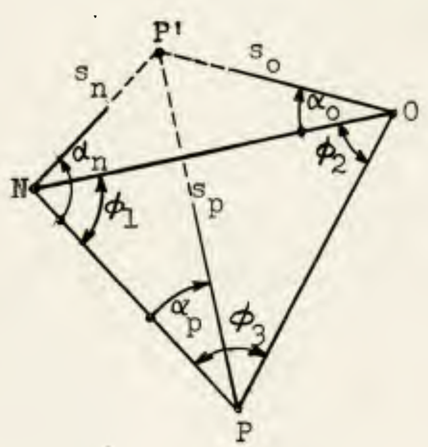
After the measured angles are corrected for systematic instrumental errors according to adjustment charts and the refraction correction is applied to the elevation angles, we have the "measured" angles:

Station	Horizontal Angle	Elevation Angle	Type of Instrument	Instrumental Accuracy (Mean Error)
N	$112^\circ 59' 45''$ from $\overline{NP}$	$80^\circ 00' 10''$	Cine-theodolite	$\pm$ 20"
P	$31^\circ 00' 00''$ from $\overline{PN}$	$72^\circ 29' 54.0''$	Ballistic Camera	$\mp$ 3"
O	$20^\circ 05'$ -- from $\overline{OP}$	--	Tracking Telescope	$\mp$ 200"

If we assign unit weight to a Cine-theodolite measurement, the following weights must be introduced:

Station	Weight
N	$p_n = 1$
P	$p_p = 50$
O	$p_o = \frac{1}{100}$

Because of the fact that we have no elevation information at station O, we have five measured quantities or 5-3 = 2 condition equations.



From the coordinates of the stations the angles ϕ_1, ϕ_2, ϕ_3 are known. They have already been corrected for spherical excess.

$$\phi_1 = 92^\circ 14' 39.1''$$
$$\phi_2 = 30^\circ 18' 56.7''$$
$$\phi_3 = 57^\circ 26' 24.2''$$
$$\Sigma = 180^\circ 00' 00.0''$$

The condition equations are, from (11) and (7):

$$\rho\left[a_1 \sin\alpha'_p \left(\tan\epsilon_n + [\mathrm{I}]\right) - a_2 \sin\alpha'_n \left(\tan\epsilon_p + [\mathrm{II}]\right) + a_3 \sin\gamma' = d'\right.$$

$$\rho\left[\sin(\phi_1 - \alpha'_n) \sin(\phi_2 + \alpha'_o) \sin\alpha'_p + \sin\alpha'_n \sin\alpha'_o \sin(\phi_3 - \alpha'_p) = D'\right.$$

$$a_1 = 1 + \frac{y_n}{r_n} = 1.00\ 240 \qquad r_n = 6363154.8 \text{ (for azimuth (NP) } -\alpha_n)$$

$a_2 = 1 + \frac{y_p}{r_p} = 1.000\ 192$ $\quad r_p = 6353492.0$ (for azimuth (PN) $+\alpha_p$)

$a_3 = \frac{\Delta y}{b} = +0.005\ 307$ $\quad$ The values for r can be taken from a table.

$\alpha'_n = \alpha_n - \frac{\Delta}{3} = 112^\circ 59' 42.7''$ $\quad \frac{\Delta}{3} = 2.3''$ can be taken from a table with the arguments α_n and α_p.

$\alpha'_p = \alpha_p - \frac{\Delta}{3} = 30^\circ 59' 57.7''$

$\gamma' = \alpha'_n + \alpha'_p = 143^\circ 59' 40.5''$

$\alpha'_o = \alpha_o = 20^\circ 05'$

The spherical excess for α_o can be neglected due to the mean error of $\pm$ 200".

	α'_n	α'_p	γ'	$a_3 \cdot \frac{\sin}{\cos} \gamma'$
sin	+ 0.920 5375	+ 0.515 0287	+ 0.587 8618	+ 0.003120
cos	- 0.391	+ 0.857	- 0.809	- 0.004

$$s_n = \frac{b \sin \alpha'_p}{\sin \gamma'} = 50\ 316.587$$

(Set of Legendre)

$$s_p = \frac{b \sin \alpha'_n}{\sin \gamma'} = 89\ 933.445$$

$(\frac{s_n}{r_n}) = 0.007\ 907$ $\quad \tan \epsilon_n = 5.672890$ $\quad \frac{1}{2} + \tan^2 \epsilon_n = 32.681\ 682$

$(\frac{s_n}{r_n})^2 = 0.000\ 063$ $\quad \tan^2 \epsilon_n = 32.181682$ $\quad \frac{5}{6} + \tan^2 \epsilon_n = 33.015\ 015$

$(\frac{s_n}{r_n})^3 = 0.000\ 0005$ $\quad \tan^4 \epsilon_n = 1035.6607$ $\quad \frac{5}{24} + \frac{7}{6} \tan^2 \epsilon_n = 37.753\ 627$

$[I] = 0.270\ 667$ $\quad \tan \epsilon_n + [I] = 5.943\ 558$

$(\frac{s_p}{r_p}) = 0.014\ 155$ $\quad \tan \epsilon_p = 3.171\ 273$ $\quad \frac{1}{2} + \tan^2 \epsilon_p = 10.556\ 973$

$(\frac{s_p}{r_p})^2 = 0.000\,200$ $\qquad \tan^2\epsilon_p = 10.056\,973$ $\qquad \frac{5}{6} + \tan^2\epsilon_n = 10.890\,306$

$(\frac{s_p}{r_p})^3 = 0.000\,003$ $\qquad \tan^4\epsilon_p = 101.1427$ $\qquad \frac{5}{24} + \frac{7}{6}\tan^2\epsilon_p = 11.941\,469$

$[II] = 0.156\,674$ $\qquad \tan\epsilon_p + [II] = 3.327\,948$

$\cos\epsilon_n = 0.174$ $\qquad \cos^2\epsilon_n = 0.030$ $\qquad d = 0.000\,867$

$\cos\epsilon_p = 0.301$ $\qquad \cos^2\epsilon_p = 0.090$ $\qquad d' = +178.83$

$A_1 = +1.30$ $\qquad A_2 = +5.09$ $\qquad A_3 = +18.63$ $\qquad A_4 = -11.09$

		Sin	Cos
$\phi_1 - \alpha'_n$	$= -20^\circ 45' 03.6''$	$-0.354\,308$	0.935
$\phi_2 + \alpha'_o$	$= +50^\circ 23' 56.7''$	0.770 503	0.537
$\phi_3 - \alpha'_p$	$= +26^\circ 26' 26.5''$	0.445 271	0.895
α'_n	$= 112^\circ 59' 42.7''$	0.920 538	-0.391
α'_p	$= 30^\circ 59' 57.7''$	0.515 029	0.857
α'_o	$= -20^\circ 05'$ --	$-0.343\,387$	0.939

$D = +.0001500$ $\qquad D' = +30.94$

$B_1 = +0.43$ $\qquad B_2 = +0.27$ $\qquad B_3 = -0.52$

and the condition equations are:

$$+1.30\, v_{\alpha_n} + 5.09\, v_{\alpha_p} + 0.\, v_{\alpha_o} + 18.63\, v_{\epsilon_n} - 11.09\, \epsilon_p + 178.83 = 0$$

$$+0.43\, v_{\alpha_n} - 0.52\, v_{\alpha_p} + 0.27\, v_{\alpha_o} + 0\, v_{\epsilon_n} + 0\, v_{\epsilon_p} + 30.94 = 0$$

$$\left[\frac{AA}{p}\right] = 351.72 \qquad \left[\frac{AB}{p}\right] = +0.51 \qquad \left[\frac{BB}{p}\right] = +7.48$$

and the normal equations:

$$\begin{array}{llll} \underline{351.72} \; K_1 + 0.51 & K_2 + 178.83 = 0 & & K_1 = -0.50 \\ \quad \underline{+ 7.48} & K_2 + \; 30.94 = 0 & & K_2 = -4.14 \end{array}$$

$$\begin{array}{lll} v_{\alpha_n} = -2.4'' & v_{\epsilon_n} = -9.3'' & \text{Check: } [pvv] = 217.7 \\ v_{\alpha_p} = +0.0 & v_{\epsilon_p} = +0.1'' & \qquad [kd] = 217.5 \\ v_{\alpha_o} = -111.8'' & - & \end{array}$$

The mean error of an observation of unit weight is $\mu = \pm \sqrt{\frac{218.2}{5-3}} = \pm 10.4''$, and the mean errors of the observation are:

$$\begin{array}{rl} \text{for station N:} & \pm \; 10.4'' \\ \text{O:} & \pm 104.0'' \\ \text{P:} & \pm \quad 1.5'' \end{array}$$

If we apply the corrections to the measured angles we obtain the adjusted angles:

$$\begin{array}{ll} \alpha_n = 112^\circ 59' 42.6'' & \epsilon_n = 80^\circ 00' 00.7'' \\ \alpha_o = \; 20^\circ 03' 08.2'' & \\ \alpha_p = \; 31^\circ 00' 00.0'' & \epsilon_p = 72^\circ 29' 54.1'' \end{array}$$

These adjusted angles represent now a rigid solution. An independent check for the adjusted angles may be obtained by substituting the adjusted values in the condition equations (6) and (9), which must be satisfied.

CONCLUSIONS

The adjustment with conditioned observations is useful for a spatial triangulation from two to four measuring stations.

The method enables us to use incomplete station measurements and to combine different instrument types by introducing weighting factors.

The accuracy with which the result is obtained depends on the number of terms carried in the formulas. The formulas are expressed in series form, thus providing a means for breaking off the numerical work at the most economical point.

The method is a rigorous adjustment. The calculated corrections for the measured angles are obtained without recourse to approximation values

of the coordinates. Information about the consistency of each of the measuring instruments is directly available by comparison of the most probable angular corrections. A calculation check is provided by $[pvv] = -[kd]$. After applying the corrections to the measured angles, the triangulation presents a rigid solution and the coordinates of a point may be calculated from any two stations. In this connection a proposal for an ellipsoidal rectangular coordinate system is outlined in another report[1].

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[1] Hellmut Schmid, "A Rectangular Ellipsoidal Coordinate System for Trajectory Measurement," BRL Report No. 748.

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